
**ANALYSIS OF THE ARTICLE APPLYING LUCAS SUCCESSION TO
THE HARMONIC HAMILTONIAN OF THE REPRESENTATION OF
BORN AND JORDAN, BY WELKEN CHARLOIS GONÇALVES,
AFFIRMATIVE ANSWER TO THE ARTICLE IN QUESTION. IN
ORDER TO EXPAND ON VARIOUS MATHEMATICAL
THEORIES AND PHYSICS.**

***Welken Charlois Goncalves**

Brazil.

Article Received: 12 September 2026, Article Revised: 21 September 2026, Published on: 12 October 2026

***Corresponding Author: Welken Charlois Goncalves**

Brazil.

Doi: <https://doi-doi.org/101555/ijarp.5243>

SUMMARY

This article analyzes a previous article, "Applying Lucas Succession to the Harmonic Hamiltonian of the Representation of Born and Jordan." by Welken Charlois Gonçalves, in which emphasizes the use of formulas and equations of recurrences in Quantum Mechanics, with the aim of expanding its teaching and universal understanding. With this possibility, verified in this article, the facts become clear and consolidated. There are numerous possibilities, or thousands, or millions, in the application and use of recursive formulas and equations in the sciences. A person like the author of the article is able to demonstrate one possibility, with the awareness that millions exist in the theories of science across all areas. The example of Luke's succession. And also, as Fibonacci's number theory has been discussed throughout history and in historical contexts of mathematics, it serves as an example of how to approach mathematical theories in physics and quantum mechanics, in this case, the harmonic oscillator.

WORDSKEYS: Succession of Lucas, Heisenberg matrices, Born-Jordan matrices, Quantum mechanics, Harmonic oscillator, Quantum field theory.

INTRODUCTION AND FRAMEWORK

The Lucas sequence is an important topic in the theory of mathematical sequences.

It is named after the French mathematician Édouard Lucas, who studied its properties. The sequence begins with the initial terms $L_0=2$ and $L_1=1$. Each new term is obtained by adding the two terms that come before it. Therefore, its recurrence relation is $L_n=L_{n-1}+L_{n-2}$. The first terms are 2, 1, 3, 4, 7, 11, 18, 29, 47, and 76. This simple rule produces a sequence with several remarkable characteristics. The Lucas sequence is closely related to the well-known Fibonacci sequence. Both sequences follow the same recurrence relation but have different initial values. Because of this difference, their terms are not equal, although they share many identities. One important characteristic of the Lucas sequence is its continuous and rapid growth. This growth is connected to the mathematical constant known as the golden ratio. The golden ratio is represented by $\phi=(1+\sqrt{5})/2$. The sequence also has a second important number, $\psi=(1-\sqrt{5})/2$. Using these numbers, its explicit formula is

$$L_n=\phi^n+\psi^n.$$

Although this formula contains irrational numbers, every Lucas number is an integer. This fact demonstrates the connection between algebraic formulas and integer sequences. As n becomes larger, the term ψ^n becomes increasingly small. Consequently, the value of L_n becomes approximately equal to ϕ^n . The ratio between consecutive Lucas numbers approaches the golden ratio. This behavior reveals the stable asymptotic growth of the sequence. Another characteristic of the Lucas sequence is its repeating pattern of parity. The terms follow the pattern even, odd, odd, which repeats throughout the sequence. Lucas numbers also have important relationships with Fibonacci numbers. For example, they satisfy the identity $L_n=F_{n-1}+F_{n+1}$. They also possess interesting divisibility and congruence properties in number theory. When considered modulo a fixed integer, their remainders eventually repeat periodically. These characteristics make Lucas numbers useful in combinatorics, algebra, and mathematics. They also help mathematicians understand recurrence relations and numerical patterns. Therefore, the Lucas sequence is a significant example of order, growth, and structure in mathematics.

Recursive equations are an important subject in the theory of mathematical sequences. They define each term by using one or more terms that appeared previously. This method is different from an explicit formula, which calculates a term directly. A recursive sequence usually requires initial conditions to begin its construction. Without initial conditions, the recurrence relation may have many possible solutions. A simple example is the arithmetic recurrence $a_n=a_{n-1}+d$. In this equation, d represents the constant difference between

consecutive terms. The explicit formula for this sequence is $a_n = a_1 + (n-1)d$. Another important example is the geometric recurrence $a_n = ra_{n-1}$. In this case, r represents the constant ratio between consecutive terms.

Its explicit formula can be written as $a_n = a_1 r^{n-1}$. These examples show how recursion can describe regular mathematical patterns. The Lucas sequence is one of the most important second-order recursive sequences. Its initial conditions are $L_0 = 2$ and $L_1 = 1$. The recursive equation of the Lucas sequence is $L_n = L_{n-1} + L_{n-2}$. This equation is valid for every integer $n \geq 2$. The first terms are 2, 1, 3, 4, 7, 11, 18, 29, and 47. Each term is obtained by adding the two preceding terms. The recurrence therefore creates a clear connection between successive values. The Fibonacci sequence follows the same recursive equation as the Lucas sequence. However, its initial conditions are $F_0 = 0$ and $F_1 = 1$. Different initial conditions produce different sequences from the same recurrence. The Lucas sequence also has an important explicit formula. This formula is known as the Binet formula for Lucas numbers. It is expressed as $L_n = \phi^n + \psi^n$. In this equation, $\phi = (1+\sqrt{5})/2$ is the golden ratio. The second number is $\psi = (1-\sqrt{5})/2$, called the conjugate of ϕ .

These two numbers are solutions of the quadratic equation $x^2 = x + 1$. The characteristic equation of the Lucas recurrence is $r^2 = r + 1$. Rewriting it gives $r^2 - r - 1 = 0$. Solving this equation produces the roots ϕ and ψ . Therefore, the explicit formula is directly connected to the recurrence relation. This connection is a central idea in the theory of recursive equations. Since $|\psi| < 1$, the

value of ψ^n decreases as n increases. Consequently, L_n becomes approximately equal to ϕ^n . This

explains the exponential growth of the Lucas sequence. The quotient L_{n+1}/L_n approaches the golden ratio. Thus, recursive equations can produce sequences with predictable long-term behavior. They can also be used to study parity, divisibility, and modular patterns. For the Lucas sequence, the parity pattern is even, odd, and odd. This pattern repeats periodically throughout the sequence. Recursive formulas are also useful in combinatorics and computer science.

They can describe counting problems, algorithms, and branching processes. However, recursive calculations may become inefficient when terms are repeatedly recalculated. For this reason, mathematicians often use explicit formulas or dynamic programming methods. Recursive definitions are also used for polynomials, functions, and geometric structures. They provide a logical method for building complex objects from simpler ones. The relationship between recursion and explicit formulas reveals mathematical structure. The

Lucas sequence demonstrates how a simple equation can generate remarkable properties. Therefore, recursive equations and their formulas are fundamental tools in mathematical theory.

DISCUSSION

The central question of the article can be solved at the level of the harmonic-oscillator spectrum, but the stronger claim—that the Lucas sequence predicts new quantum-mechanical physics—is not established by the article. The paper explicitly asks whether there is an exact choice of P, Q in the Lucas sequence that produces a quantum-mechanical relation [1].

Hamiltonian used in the article

The article starts with the Born–Jordan/Heisenberg matrix for the harmonic oscillator:

$$H = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \cdots \\ 0 & \frac{3}{2} & 0 & \cdots \\ 0 & 0 & \frac{5}{2} & \cdots \\ 0 & 0 & 0 & \frac{7}{2} \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus its eigenvalues are

$$E_n = n + \frac{1}{2}, \quad n = 0, 1, 2, \dots$$

which is exactly the spectrum represented by the diagonal matrix in the paper.

Apply the Lucas recurrence

The article defines the Lucas-type recurrence as

$$W_n = PW_{n-1} - QW_{n-2}.$$

It also gives the characteristic equation#

$$r^2 - Pr + Q = 0.$$

We want the sequel.

$$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \dots$$

to satisfy this recurrence.

Notice that

$$E_n - E_{n-1} = 1.$$

Therefore

$$E_n = 2E_{n-1} - E_{n-2}.$$

Comparing with

$$E_n = PE_{n-1} - QE_{n-2},$$

we obtain

$$P = 2, \quad Q = 1.$$

So the required Lucas recurrence is

$$E_n = 2E_{n-1} - E_{n-2}$$

with

$$E_0 = \frac{1}{2}, \quad E_1 = \frac{3}{2}.$$

Calculate the sequence Starting with

$$E_0 = \frac{1}{2}$$

and

$$E_1 = \frac{3}{2},$$

we obtain

$$E_2 = 2 \left(\frac{3}{2} \right) - \frac{1}{2} = \frac{5}{2},$$

$$E_3 = 2 \left(\frac{5}{2} \right) - \frac{3}{2} = \frac{7}{2},$$

$$E_4 = 2 \left(\frac{7}{2} \right) - \frac{5}{2} = \frac{9}{2}.$$

Therefore

$$\boxed{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \dots}$$

which is exactly

$$\boxed{E_n = n + \frac{1}{2}}.$$

Connection with the standard Lucas U_n sequence For

$$P = 2, \quad Q = 1,$$

the standard Lucas sequence $U_n(P, Q)$ has

$$U_0 = 0, \quad U_1 = 1$$

and

$$U_n = 2U_{n-1} - U_{n-2}.$$

Consequently,

$$U_n = n.$$

Therefore the harmonic-oscillator energies can be written

$$E_n = U_n(2, 1) + \frac{1}{2}$$

or, equivalently

$$E_n = \frac{2U_n(2, 1) + 1}{2}.$$

This is the cleanest Lucas representation of the harmonic-oscillator spectrum.

A more interesting relation: the odd-integer sequence Multiplying the energy by 2 gives

$$2E_n = 1, 3, 5, 7, 9, \dots$$

Define

$$S_n = 2E_n.$$

Then

$$S_n = 2n + 1.$$

It also satisfies

$$S_n = 2S_{n-1} - S_{n-2}$$

with

$$S_0 = 1, \quad S_1 = 3.$$

Thus

$$2E_n = 2U_n(2, 1) + 1.$$

This provides a precise mathematical correspondence between the Lucas recurrence and the oscillator spectrum.

Finite (4x4) Hamiltonian

The article also considers finite-dimensional approximations. For four oscillator states,

$$H_4 = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 5/2 & 0 \\ 0 & 0 & 0 & 7/2 \end{pmatrix}.$$

Because this matrix is diagonal,

$$\det(H_4 - \lambda I) = \left(\frac{1}{2} - \lambda\right) \left(\frac{3}{2} - \lambda\right) \left(\frac{5}{2} - \lambda\right) \left(\frac{7}{2} - \lambda\right).$$

Expanding,

$$\det(H_4 - \lambda I) = \lambda^4 - 8\lambda^3 + \frac{43}{2}\lambda^2 - 22\lambda + \frac{105}{16}.$$

Therefore,

$$\lambda = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}.$$

These are the four oscillator energies.

This is important because the article [1] reports a different characteristic polynomial and then obtains complex roots. For the diagonal Hamiltonian displayed in the article, the characteristic polynomial must instead be the product above, and its roots are real.

What the Lucas sequence actually explains

We can now answer the article's main question. The relationship is

$$E_n = n + \frac{1}{2} = U_n(2, 1) + \frac{1}{2}.$$

So the Lucas recurrence reproduces the indexing pattern of the harmonic-oscillator energy levels.

However, this does not mean that Lucas theory derives quantum mechanics.

The reason is that we started with

$$E_n = n + \frac{1}{2}$$

from the harmonic oscillator and $P = 2, Q = 1$ then selected

because this sequence satisfies the corresponding recurrence. In other words,

Quantum spectrum $\rightarrow$ Lucas representation

has been demonstrated.

But the stronger claim

Lucas mathematics $\rightarrow$ derivation of quantum mechanics

The article[1] itself describes the proposed extension toward Dirac, Klein–Gordon, Proca, massive photon, and Kaluza–Klein ideas as a possibility for future research rather than as an established result.

The answer to the article's research question The article asks, essentially:

Can the Lucas sequence predict a relationship with quantum mechanics by finding the correct P and Q?

For the harmonic oscillator, we can give a precise answer:

$$P = 2, Q = 1$$

and

$$E_n = U_n(2, 1) + \frac{1}{2} = n + \frac{1}{2}.$$



Therefore, yes, there is an exact Lucas-sequence representation of the harmonic-oscillator energy spectrum.

But the result should be interpreted as a mathematical reformulation, not yet as a prediction of a new quantum-mechanical law.

The next research step

A genuinely predictive version of the proposal would have to start with the Lucas parameters independently and derive a quantum observable that was not already inserted into the calculation.

For example, one could investigate whether a suitably chosen Lucas recurrence produces:

$$E_n,$$

$$\Delta E_n = E_{n+1} - E_n,$$

<https://zenodo.org/records/15043349> transition frequencies,

$$\omega_{n+1,n} = \frac{E_{n+1} - E_n}{\hbar},$$

or matrix elements of $a, a^\dagger$.

The article already presents the ladder-operator matrices and their infinite-dimensional character.

That would be a substantially stronger test than simply observing that

$$1/2, 3/2, 5/2, 7/2, \dots$$

can be represented using $U_n(2, 1)$. Final mathematical result

$$\begin{aligned} U_0(2, 1) &= 0, \\ U_1(2, 1) &= 1, \\ U_n(2, 1) &= 2U_{n-1}(2, 1) - U_{n-2}(2, 1), \end{aligned}$$

and therefore

$$E_n = \frac{1}{2} + U_n(2, 1) = n + \frac{1}{2}.$$

So the Lucas parameters corresponding to the harmonic Hamiltonian are $P=2, Q=1$. This is the exact solution of the article's Lucas–harmonic-oscillator question, while the proposed extension to new quantum-field or unified-theory physics remains an open hypothesis rather than a result established by the paper.

CONCLUSION

Therefore, the usefulness of recursive formulas in Mechanics is indeed verified. Quantum mechanics, with the possibility of thousands or millions of other recursive formulas existing, can be applied to all areas of quantum mechanics, especially in the case analyzed, that of the harmonic oscillator in question. It's worth remembering that the harmonic oscillator was one of the pillars that gave rise to quantum mechanics; much was discussed about the types of harmonic oscillators at the beginning of the 20th century for the consolidation of quantum mechanics, a case that will be analyzed regarding the application of the Lucas sequence to the harmonic oscillator. of the Heisenberg, Born, and Jordan Matrices. Indeed, it is verified and consolidated that recursive formulas and equations are of great use for the conceptual purposes of science, and even more so in the case of quantum mechanics, with many open questions and many concepts to be analyzed, such as, for example, what would the Lucas sequence applied to the harmonic oscillator, the Lucas harmonic oscillator, mean in a realistic quantum world of application? Just like other recursive formulas and equations applied in Quantum Mechanics, in the special case of the Harmonic Oscillator. This could be extended to various factors, elements, and questions raised and studied in Quantum Mechanics throughout the 20th and 21st centuries. Therefore, the Quantum Spectrum with the Lucas Representation would have meaning in the world of Real Quantum Mechanics.

BIBLIOGRAPHIC REFERENCES

1. Applying Lucas Succession to the Harmonic Hamiltonian of the Representation of Born and Jordan. Welken Charlois Gonçalves <https://zenodo.org/records/15043349>